## **7. Forced oscillation of the string**

Our general aim is the analysis of the movement of the bounded string under the exterior force. At first, we consider the string without exterior force for the case, where the law of movement of the string ends is known. This problem is described by the vibrating string equation with non-homogeneous boundary conditions. Changing the unknown function, we transform this problem to the non-homogeneous equation with homogeneous boundary conditions. The analogical system is the mathematical model of the oscillation of the string under the exterior force. This problem is solved by Fourier method. The solution of this system is found as a Fourier series. We find the corresponding Fourier coefficients using the given initial conditions. The oscillation of the string under easy enough exterior force is considered as example.

### **7.1. String with known law of movement of ends**

Consider the movement of the bounded string. We have the vibrating string equation

*utt = a*2*uxx*, 0 < *x* < *L*, *t* > 0. (7.1)

Let the ends of the string move, i.e. we have the non-homogeneous boundary conditions

*u*(0,*t*) = *p*(*t*), *u*(*L*,*t*) = *q*(*t*), *t* > 0, (7.2)

where the functions *p* and *q* are known. The initial state and the initial velocity of the string are given s before

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*. (7.3)

The system (7.1) – (7.3) is called the ***non-homogeneous first boundary problem*** for the vibrating string equation.

We cannot to use the method of separation of variables because of the boundary conditions. Try to find the solution of this problem as the sum

*u*(*x*,*t*) = *v*(*x*,*t*) + *w*(*x*,*t*), (7.4)

where we choose the function *w* such that it satisfies the boundary conditions (7.2). The easiest variable function is linear. Then we choose the function *w* by the formula

*w*(*x*,*t*) = *α*(*t*)+*β*(*t*)*x*.

We choose the functions *α* and *β* for obtaining the equalities (7.2). We have

*w*(0,*t*) = *α*(*t*) = *p*(*t*),

*w*(*L*,*t*) = *α*(*t*)+*β*(*t*)*L = q*(*t*).

Then we determine

*α*(*t*) = *p*(*t*), *β*(*t*) *=* [*q*(*t*) – *p*(*t*)]*x*/*L*.

Now we obtain

*w*(*x*,*t*) *= p*(*t*) *+* [*q*(*t*) – *p*(*t*)]*x*/*L*. (7.5)

Using the formula (7.5), put the function *u* from the equality (7.4) to the equation (7.1). We have

*vtt* + *p"*(*t*) *+* [*q"*(*t*) – *p"*(*t*)]*x*/*L*] *= a*2*vxx*.

Therefore, we have the equality

*vtt = a*2*vxx* + *f*(*x*,*t*), (7.6)

where

*f*(*x*,*t*) = –{*p"*(*t*) *+* [*q"*(*t*) – *p"*(*t*)]*x*/*L*]}.

Now we put the function *u* from the equality (7.4) to the equation (7.3). We get

*v*(*x*,0) + *p*(0) *+* [*q*(0) – *p*(0)]*x*/*L* = *ϕ*(*x*),

*vt*(*x*,0) + *p*'(0) *+* [*q*'(0) – *p*'(0)]*x*/*L* = *ψ*(*x*).

Then we obtain the initial conditions for the function *v*

*v*(*x*,0) = *ϕ*1(*x*), *vt*(*x*,0) = *ψ*1(*x*), (7.7)

where

*ϕ*1(*x*) = *ϕ*(*x*) – *p*(0)–[*q*(0) – *p*(0)]*x*/*L*,

*ψ*1(*x*) = *ψ*(*x*) – *p*'(0)–[*q*'(0) – *p*'(0)]*x*/*L.*

By choosing of the function *w* the boundary conditions for the function *v* is homogeneous

*v*(0,*t*) =0, *v*(*L*,*t*) = 0. (7.8)

Thus, the given problem (7.1) – (7.3) is transformed to the non-homogeneous vibrating string equation (7.6) with initial conditions (7.7) and homogeneous boundary conditions (7.8). If we find the solution of this problem, then we will find the solution of the initial problem (7.1) – (7.3) by the formula (7.4). Note that the problem (7.6) – (7.8) has a direct physical sense.

### **7.2. Mathematical model of the string forced oscillation**

Consider the movement of the bounded string of length *L* under an exterior force. The phenomenon is by the non-homogeneous vibrating string equation

*utt = a*2*uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0, (7.9)

where the function *f* characterizes the influence of the exterior force. Suppose the ends of the string are fixed. Then we have the homogeneous first order boundary conditions

*u*(0,*t*) = 0, *u*(*L*,*t*) = 0, *t* > 0. (7.10)

The initial state *ϕ* =*ϕ*(*x*) and the initial velocity*ψ* =*ψ*(*x*) of the string are given. Then we have the initial conditions

*u*(*x*,0) = *ϕ*(*x*), *ut*(*x*,0) = *ψ*(*x*), 0 < *x* < *L*. (7.11)

We have the first homogeneous boundary problem for the non-homogeneous vibrating string equation.

### **7.3. Fourier method**

We know that the corresponding boundary problem for the homogeneous boundary problem has the solution that is represented as a sinus Fourier series. We try to find the solution of the problem (7.9) – (7.11) in the analogical form, i.e.

 (7.12)

where the functions *uk*that are Fourier coefficients are unknown. Note that for all *uk* the function *u* of the formula (7.12) satisfies the boundary conditions (7.10). We try to find the functions *uk*such that the function *u* determined by the formula (7.12) satisfies the equation (7.9) and the boundary conditions (7.11). This is the idea of the ***Fourier method.***

Put the function *u* from the equality (7.12) to the equation (7.9). We get



Multiply this equality by the function  and integrate the result by *x* from 0 to *L.* We have

 (7.13)

Find the integral



Determine



If  we can find



Finally, for *k = n* we get



Putting the results to the formula (7.13), we obtain



Determine the functions

 (7.14)

There are sinus Fourier coefficients of the function *f.* Then the previous formula can be transformed to the equality

 (7.15)

If the function *un* satisfies the equation (7.15), then the function *u* determined by the formula (7.12) is the solution of the non-homogeneous vibrating string equation (7.9).

Put the function *u* from the formula (7.12) to the initial conditions (7.11). We get





Multiply these equalities by the function  and integrate the result by *x* from 0 to *L.* We have





Using the previous formulas of integrals, determine

 

Denote

  (7.16)

Then we have

  (7.17)

If the function *un* satisfies the conditions (7.17), then the function *u* determined by the formula (7.12) satisfies the initial conditions (7.11).

Thus, it is necessary to solve the problem (7.15), (7.17) for determining the Fourier coefficient *un*.

### **7.4. Finding the solution of the problem**

Determine the general solution of the non-homogeneous equation (7.14), using the ***method of variation of parameters***. Consider, at first, the corresponding homogeneous equation



where  Its general solution is determined by the formula



where *α* and *β* are arbitrary constant. By the method of variation of parameters the general solution of initial non-homogeneous equation is determined by analogical formula with variable parameters

 (7.17)

where the derivatives of *α* and *β* satisfies the following system



Thus, we have the system of linear algebraic equations

 (7.19)

Multiply the first equality by  and the second by . After summing, we get



Now we multiply the first equality (7.19) by  and the second by . After differing, we find



After integration we have





Put the results to the formula (7.17). We get



Using the known trigonometric formula, we find

 (7.20)

Put *t =* 0 at the formula (7.18) and use the first initial condition (7.17). We have



Now differentiate the equality (7.17). Determine



For *t =* 0 we have



We determined before that  Then we find



Put the result to the formula (7.20). Thus, the solution of the problem (7.15), (7.17) is



Using the formula (7.12), we find the solution of the initial boundary problem

 (7.21)

Thus, for determining the solution of the non-homogeneous vibrating string equation (7.9) with boundary conditions (7.10) and the initial conditions (7.11), it is necessary to find the Fourier coefficients  by the formulas (7.14), (7.16) and use the formula (7.21).

Remark. This method is applicable for the free ends of the string. In this situation, the solution of the problem is determined as a cosine Fourier series.

### **7.5. Forced oscillation of the string**

Consider the partial case of the problem (7.9) – (7.11). Let us analyze the string of the length *L=π* with coefficient *a =* 1 and the exterior force determined by the function *f*(*x*,*t*) = sin *x*. Then we have the non-homogeneous vibrating string equation

*utt = uxx* + sin *x*, 0 < *x* < *π*, *t* > 0. (7.22)

The ends of the string are fixed. Then we have the boundary conditions

*ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (7.23)

Suppose the initial state and the initial velocity of the string are zero. Then we have the initial conditions

*u*(*x*,0) = 0, *ut*(*x*,0) = 0, 0 < *x* < *L*. (7.24)

Using the formula (7.20), determine the solution of the problem (7.22) – (7.24) by the formula

 (7.25)

By the formulas (7.16) and initial conditions (7.24), determine



Find the coefficients *fk* by the formula (7.14). We have



If *k* > 1, we determine



For *k =* 1 we find





Using the formula of cosine product, we have

.

Find



Calculate the second integral



The value of the first integral depends from the number *k.* If *k*>1, we find



Put the results to the formula (7.24). Thus, the solution of the problem (7.21) – (7.23) is

 (7.26)

Give the physical interpretation of this result. At the initial time, the string is in the state of the state of equilibrium and has zero velocity. The string moves by the force that does not depend from time. For any fixed time *t* the string has the form of sinus. However, the position of the string changes with respect to the time. Any fixed point *x* oscillates with period 2*π.* Its swing amplitude depends from the point *x*. This is equal to sin *x*. For example, the middle of the string that is the point *x =* *π*/2 has the position 0 at the time *t =* 0, √2/2 for *t = π*/4, 1 for *t = π*/2, √2/2 for *t =* 3*π*/4, 0 for *t = π*, -√2/2 for *t =* 5*π*/4, -1 for *t = π*, -√2/2 for *t =* 7*π*/4, 0 for *t =* 2*π*, etc., see the following figure.



Figure 7.1. Oscillation of the string.

### **Conclusions**

* The homogeneous vibrating string equation with non-homogeneous boundary conditions can be transformed to the non-homogeneous vibrating string equation with homogeneous boundary conditions.
* The movement of the string under exterior force with fixed ends is described by the boundary problem for the non-homogeneous vibrating string equation.
* The solution of the problem is represented as a sinus Fourier series by the Fourier method.
* The Fourier coefficients of solution depend from the time and satisfy non-homogeneous second order ordinary differential equations with initial conditions.
* The parameters of the obtained system are Fourier coefficients of the force function and initial states.
* The solution of the non-homogeneous ordinary differential equations are found by the method of variation of parameters.
* The Fourier coefficients of this representation are determined by the initial conditions of the considered problem.
* If the ends of string applicable are free, then the solution of the problem is determined as a cosine Fourier series.
* The oscillation of the string under exterior force can be analyzed as an application of these results.

### **Task**. **Oscillation of the string under exterior force**

Consider the movement of the body under string exterior force characterized by the given function *f.* This is described by non-homogeneous vibrating string equation

*utt = a2 uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0.

Suppose the string is in the state of equilibrium and has zero velocity at the initial time. Then we have the initial conditions

*u*(*x*,0) = 0, *ut*(*x*,0) = 0, 0 < *x* < *L*.

The ends of the string can be free or fixed, i.e. we have one of the following boundary conditions

*u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0; (\*)

*ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (\*\*)

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary condition | *L* | *a* | *f* |
| 1 | \* | 1 | 2 | –sin π*x* |
| 2 | \*\* | π | 1 | cos *x* |
| 3 | \* | π | ½ | –2 sin *x* |
| 4 | \* | 1 | 3 | sin 2π*x* |
| 5 | \*\* | π | ½ | cos 2*x* |
| 6 | \*\* | 1 | 1 | –cos π*x* |
| 7 | \*\* | 1 | 2 | cos 2π*x* |
| 8 | \* | π | ½ | sin 2*x* |

Task:

1. Determine the solution of the problem as sinus Fourier series for the boundary conditions (\*) and cosine Fourier series for the boundary conditions (\*\*).
2. Find the Fourier coefficient of the parameters of the system.
3. Solves ordinary differential equations with respect to the Fourier coefficients of the solution of the problem.
4. Check that this is, in reality the solution of the boundary problem.
5. Show the graph (position of the string for the different time points).
6. Give the physical interpretation of the results.